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**LETTER TO THE EDITOR**

# Classification of invariant solutions of the Boltzmann equation

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**Abstract.** An isomorphism of the Lie algebras $L_{11}$ admissible by the full Boltzmann kinetic equation with an arbitrary differential cross section and by the Euler gas dynamics system of equations with a general state equation is set up. The similarity is also proved between extended algebras $L_{12}$ admissible by the same equations for specified power-like intermolecular potentials and for polytropic gas. This allows the solution of the problem of classification of the full Boltzmann equation invariant H-solutions using an optimal system of subalgebras known for the Euler system. Representations of essentially different H-solutions of the spatially inhomogeneous Boltzmann equation with one and two independent invariant variables in the explicit form are obtained on this basis.

## *Introduction*

Earlier symmetries and invariant H-solutions of the full Boltzmann equation (BE) of the kinetic gas theory were studied by ad hoc methods. In most of these studies a form of admissible transformation was postulated *a priori* (see [1, 2] for a review). At present, as the result, it was shown by the authors of [3–5] and others that the full BE admits the 11-parameter Lie group $G_{11}$ of point transformations. Some extensions of the $G_{11}$ group for certain intermolecular potentials are also known.

By virtue of the heuristic approach the question about the completeness of the found admissible groups has remained open. In spite of this fact it would be very useful to carry out a classification of the set of H-solutions for a constructive description of the BE solutions invariant with respect to these Lie groups.

This classification allows separating the set of H-solutions into non-intersecting essentially different classes, obtaining the representations of the H-solutions for different classes and reducing the full BE to corresponding factor equations. The classification demands a construction of an optimal system of subgroups (subalgebras) of an admissible Lie group (algebra) [6, 7]. The general algorithms for constructing such systems are known in the group theory. However, their realization for large dimension groups such as $G_{11}$ requires very cumbersome calculations. For example, one can point to the Euler gas dynamics (EGD) system of equations that admits Lie groups of similar dimensions for different state equations. Despite the fact that these complete admissible groups were obtained in the 1970s [6] their optimal systems were only calculated in the last 3–5 years [8–10].

In this letter an isomorphism of Lie groups (algebras) admissible by the full BE and EGD-system is set up. The proved isomorphism allows using the optimal systems of subalgebras

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obtained for the EGD-system in the papers cited above for the classification of invariant solutions of the full BE. Representations of the classes of H-solutions of the BE with one and two independent invariant variables are obtained in explicit forms.

*An admissible Lie algebra*

The full BE describing the evolution of the distribution function $f(t, \boldsymbol{x}, \boldsymbol{v})$ in the product space $R_+^1 \times R_{\boldsymbol{x}}^3 \times R_{\boldsymbol{v}}^3$ is [11]

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{x}} = I(f, f) \tag{1}$$

$$J(f, f) = \int \mathrm{d}\boldsymbol{w}\, \mathrm{d}\boldsymbol{n}\, g\sigma\left(g, \frac{g\boldsymbol{n}}{g}\right)[f(\boldsymbol{v}^*)f(\boldsymbol{w}^*) - f(\boldsymbol{v})f(\boldsymbol{w})] \tag{2}$$

$$\boldsymbol{v}^* = \tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{w} + g\boldsymbol{n}) \qquad \boldsymbol{w}^* = \tfrac{1}{2}(\boldsymbol{v} + \boldsymbol{w} - g\boldsymbol{n}) \qquad \boldsymbol{g} = \boldsymbol{v} - \boldsymbol{w}$$

$$g = |\boldsymbol{g}| \qquad |\boldsymbol{n}| = 1.$$

Here $t \in R_+^1$ is time, $\boldsymbol{x} = (x, y, z) \in R_{\boldsymbol{x}}^3$ is space variable, $\boldsymbol{v} = (u, v, w) \in R_{\boldsymbol{v}}^3$ is molecular velocity; $\sigma(g, g\boldsymbol{n}/g)$ is a differential scattering cross section. For power intermolecular potentials $U(r) \propto r^{-(\nu-1)}(\nu > 2)$ there is $\sigma = g^\gamma \sigma_0(g\boldsymbol{n}/g)$, $\gamma = (\nu - 5)(\nu - 1)^{-1}$. A limit $\nu \to \infty$ corresponds to hard sphere molecules.

In [4] the admissible Lie group $G(T_a)$ of point transformations $T_a$ of the BE (1) was sought in the form

$$\begin{aligned} &f = \varphi(t, \boldsymbol{x}, \boldsymbol{v}; a)f' \qquad t' = \tau(t, \boldsymbol{x}; a) \\ &\boldsymbol{x}' = \boldsymbol{h}(t, \boldsymbol{x}; a) \qquad \boldsymbol{v}' = B(t, \boldsymbol{x}; a)\boldsymbol{v} + \boldsymbol{b}(t, \boldsymbol{x}; a) \end{aligned} \tag{3}$$

where $a$ is a group parameter, $B$ is some $3 \times 3$ matrix. A feature of the group is that the nonlinear integral collision operator has the following generalized 'scaling' property:

$$J(f', f') = \psi(t', \boldsymbol{x}', \boldsymbol{v}')J(f, f).$$

Unknown functions in (3) were found from the main property of an admissible Lie group: the BE (1) admits a Lie group $G(T_a)$ if for each $a$ a transformation (2) converts any solution of the BE into some solution of the same equation. After substituting (3) into (1) and employing the splitting-up method [6] we have obtained the admissible Lie group $G(T_a)$ in the explicit form. Here we first of all are interested in the Lie algebra corresponding to the found group $G(T_a)$. For arbitrary cross section $\sigma$ there is a Lie algebra $L_{11}(X)$ with the basis of infinitesimal generators:

$$\begin{aligned} &X_1 = \partial_x \qquad X_2 = \partial_y \qquad X_3 = \partial_z \qquad X_4 = t\partial_x + \partial_u \\ &X_5 = t\partial_y + \partial_v \qquad X_6 = t\partial_z + \partial_w \qquad X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v \\ &X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w \qquad X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u \\ &X_{10} = \partial_t \qquad X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z - f\partial_f. \end{aligned}$$

For the power intermolecular potentials there is an extension of Lie algebra $L_{11}(X)$ to an algebra $L_{12}(X)$ by the generator $X_{12} = t\partial_t - u\partial_u - v\partial_v - w\partial_w + (\gamma + 2)f\partial_f$. And for the special case $\gamma = -1$ there is one more generator $X_{13} = t^2\partial_t + t\boldsymbol{x}\partial_{\boldsymbol{x}} + (\boldsymbol{x} - t\boldsymbol{v})\partial_{\boldsymbol{v}}$, which corresponds to a projective transformation [6]. An action of the derivative $\partial_f$ in the generators $X_{11}$ and $X_{12}$ onto integral operator (2) has to be considered as the Freschet derivative.

**Remark 1.** *In [3] the Lie subalgebra with the generators $X_1, X_2, X_3, X_4, X_5, X_6, X_{10}$ was originally calculated for the Bhatnagar–Gross–Krook kinetic equation [11]. Then it was* *directly verified that these generators are admissible by the BE (1). Furthermore in [3] the generator $X_{12}$ was presented with reference to Nikol'skii (as a private communication). A mutual relation between $X_{13}$ and a point transformation of the BE for $\gamma = -1$ found by Nikol'skii [13] was pointed out in [12].*

**Remark 2.** *In [5, 14] a Lie group with generators as presented here was declared to be a full (complete) Lie group admissible by the full BE (1). However, calculations were practically carried out in a similar ad hoc approach as outlined above. We must emphasize that a rigorous proof of completness of an admissible group can only be given by deriving a general solution of the determining equations for coefficients of generators [6, 7]. For some kinetic equations including the BE with higher symmetry such proofs were presented in [15, 16]. The similar proof for a multi-dimensional $G_{11}$ group has yet to be performed.*

### *Classification of subalgebras*

Here we present a classification of all H-solutions invariant with respect to the Lie group $G_{11}$ of the BE (1). The classification subdivides a set of H-solutions into equivalence (similarity) classes. Any two H-solutions, $f_1$ and $f_2$, are elements of the same equivalence class if there exists a transformation $T_a \in G(T_a)$ such that $f_2 = T_a f_1$. Otherwise, $f_1$, $f_2$ belong to different classes and they are called essentially different H-solutions. A list of all essentially different H-solutions (one representative from each class) is an optimal system of invariant solutions that defines the searched classification. To obtain this list an *optimal system* $\Theta_L = \{N\}$ of subalgebras of the admissible Lie algebra $L$ is constructed [6, 7]. The $\Theta_L$ is a maximal set of the subalgebras $N \subset L$, any pair of which is not similar with respect to inner automorphisms of the algebra $L$. For low-dimensional algebras the calculations of $\Theta_L$ are sufficiently simple. Optimal systems for kinetic equations with high symmetry were obtained in [3, 15, 16]. But as the dimension of $L$ increases computational difficulties grow multifold.

However, for the admissible Lie algebra $L_{11}(X)$ a remarkable circumstance arises that allows us to avoid tedious calculations.

**Theorem 1.** *The Lie algebra $L_{11}(X)$ admissible by the full BE is isomorphic to the Lie algebra $L_{11}(Y)$ admissible by the EGD-system.*

**Proof.** The EGD-system is written as follows [6]:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\boldsymbol{u} = 0 \qquad \rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p = 0 \qquad \frac{\mathrm{d}p}{\mathrm{d}t} + A(\rho, p)\nabla\boldsymbol{u} = 0 \tag{4}$$

where $\rho$, $p$ are density and pressure of a gas, $\nabla$ is a nabla operator, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \nabla\boldsymbol{v}$. As above, $t \in R^1_+$, $\boldsymbol{x} = (x, y, z) \in R^3_{\boldsymbol{x}}$, $\boldsymbol{v} = (u, v, w) \in R^3_{\boldsymbol{v}}$, but now $\boldsymbol{v}$ is a vector of gas macroscopic velocity.

For an arbitrary state function $A(p, \rho)$ system (4) admits an 11-parameter Lie group of transformations [6] with the generators:

$$\begin{aligned}
&Y_1 = \partial_x \qquad Y_2 = \partial_y \qquad Y_3 = \partial_z \qquad Y_4 = t\partial_x + \partial_u \qquad Y_5 = t\partial_y + \partial_v \\
&Y_6 = t\partial_z + \partial_w \qquad Y_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v \\
&Y_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w \qquad Y_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u \\
&Y_{10} = \partial_t \qquad Y_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.
\end{aligned}$$

Let $Q(X) = Y$ be a mapping of $L_{11}(X)$ onto $L_{11}(Y)$, defined by one-to-one correspondence $Q(X_k) = Y_k$, $k = 1, \ldots, 11$. It is directly verified that $Q$ saves the commutators

$$Q([Y_k, Y_j]) = [Q(Y_k), Q(Y_j)] \qquad j, k = 1, 2, \ldots, 11 \tag{5}$$

where $[A, B] = AB - BA$. This means that $Q$ is an isomorphism and the Lie algebras $L_{11}(X)$ and $L_{11}(Y)$ are isomorphic. □

**Remark 3.** *If a state function $A(p, \rho) = \kappa\rho$ (polytropic gas), then there exists an extension of algebra $L_{11}(Y)$ by two additional generators:*

$$Y_{12} = t\partial_t - u\partial_u - v\partial_v - w\partial_w + 2\rho\partial_\rho \qquad Y_{13} = \rho\partial_\rho + p\partial_p$$

*up to a Lie algebra $L_{13}(Y) = L_{12}(Y) \oplus \{Y_{13}\}$. It is analogously proved that the Lie algebra $L_{12}(X)$ is isomorphic to the subalgebra $L_{12}(Y)$.*

**Remark 4.** *In the case of monoatomic gas $\kappa = (n + 2)/n$ ($n$ is a flow dimension) the EGD–system (4) admits one more generator:*

$$Y_{14} = t^2\partial_t + t\boldsymbol{x}\partial_{\boldsymbol{x}} + (\boldsymbol{x} - t\boldsymbol{v})\partial_{\boldsymbol{v}} - nt\rho\partial_\rho - (n + 2)p\partial_p.$$

*A connection between generators $X_{13}$ and $Y_{14}$ was noted in [12,17].*

By virtue of the proven isomorphism of the Lie algebras $L_{11}(X)$ and $L_{11}(Y)$ their optimal system of subalgebras are also isomorphic. Really the more powerful proposition is justified.

*Consequence.* For classifying and constructing essentially different H-solutions of the BE (1) one can immediately use the optimal system of subalgebras constucted for the EGD-system (4) in [8].

Indeed, it is known [6,7] that in practice a construction of an optimal system of subalgebras of a given Lie algebra is completely defined by a table of commutators of basic generators. It follows from (5) that the tables of the commutators of both algebras $L_{11}(X)$ and $L_{11}(Y)$ coincide up to notations. This proves the consequence.

**Remark 5.** *Generator $Y_{13}$ is a centre of the Lie algebra $L_{13}(Y)$: $[Y_i, Y_{13}] = 0$, $i = 1, \ldots, 12$. Taking into account remark 3 and relations (5) means that for classification of subalgebras of $L_{12}(X)$ we can use the optimal system of subalgebra $L_{12}(Y)$ that was constructed in [9].*

**Table 1.** Representations of H-solutions with one independent invariant variable.

| No | Representation | Index |
|---|---|---|
| 1 | $e^{\varepsilon\theta}\varphi(q)$ | (6.3) |
| 2 | $t^{-1}\varphi(W^2 + (V - rt^{-1})^2)$ | (6.4) |
| 3 | $t^{-1}\varphi(q)$ | (6.5) |
| 4 | $t^{-1}\varphi(u - xt^{-1})$ | (6.7) |
| 5 | $t^{-1}\varphi(u - \varepsilon\, ln\, t)$ | (6.20) |
| 6 | $\varphi(t)$ | (6.14) |
| 7 | $\varphi(u)$ | (6.11) |
| 8 | $\varphi(u - t)$ | (6.23) |
| 9 | $e^{\varepsilon\theta}\varphi(w)$, $\varepsilon \neq 0$ | (6.18) |
| 10 | $t^{-1}\varphi(q/t)$ | (7.2) |
| 11 | $x^{-1}\varphi(u)$ | (7.3) |

**Table 2.** Representations of H-solutions with two independent invariant variables.

| No | Representation | Index |
|---|---|---|
| 1 | $r^{-1}\varphi(U, q)$ | S (5.1) |
| 2 | $r^{-1}\varphi(V, W)$ | C (5.2) |
| 3 | $x^{-1}\varphi(u, q)$ | (5.3) |
| 4 | $\mathrm{e}^{-\alpha\theta}\varphi(u-\beta\theta, q)$ | C (5.4) |
| 5 | $t^{-1}\varphi(u-x/t, (v-y/t)^2+(w-z/t)^2)$ | (5.5) |
| 6 | $t^{-1}\varphi(u-x/t, q)$ | (5.6) |
| 7 | $t^{-1}\varphi(u-\beta\ln t+\alpha\arcsin((v-y/t)/q), q)$ <br> $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$ | (5.7) |
| 8 | $t^{-1}\varphi(u-\beta\ln t+\alpha\arcsin(v/q), q)$ | (5.8) |
| 9 | $\varphi(t, q)$, $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$, $\alpha=0$ | (5.9) |
| 10 | $t^{-1}\varphi(q, \arcsin((v-y/t)/q)+\alpha^{-1}\ln t)$, $\alpha\neq 0$ <br> $q=\sqrt{(v-y/t)^2+(w-z/t)^2}$ | (5.9) |
| 11 | $t^{-1}\varphi(x/t, u-\alpha^{-1}\beta\ln t)$ | (5.10) |
| 12 | $t^{-1}\varphi(\arcsin(v/q)-\alpha^{-1}\ln t, q)$ | (5.11) |
| 13 | $\varphi(u+\alpha\arcsin(v/q)-t, q)$ | (5.12) |
| 14 | $\varphi(t, u-x/t)$ | (5.13) |
| 15 | $\varphi(t, q)$ | (5.15) |
| 16 | $\varphi(t, q)$, $q=(v-(yt+z)/(1+t^2))^2+(w+(y-zt)/(1+t^2))^2$ | (5.16) |
| 17 | $\varphi(t, u)$ | (5.17) |
| 18 | $\varphi(x-t^2/2, u-t)$ | (5.18) |
| 19 | $\varphi(x, u)$ | (5.19) |
| 20 | $\varphi(q, \arcsin(v/q)+t)$ | (5.20) |
| 21 | $x^{-1}\varphi(u, w-\beta\ln x)$ | (5.21) |
| 22 | $\mathrm{e}^{-\beta u}\varphi(v, w)$ | (5.22) |
| 23 | $t^{-1}\varphi(v-y/t, w-z/t)$ | (5.24) |
| 24 | $t^{-1}\varphi(u-x/t, v-\alpha^{-1}(x-\beta\ln t))$ | (5.25) |
| 25 | $t^{-1}\varphi(x/t-\beta\ln t, u-\beta\ln t)$ | (5.26) |
| 26 | $t^{-1}\varphi(u-x/t, v-\beta\ln t)$ | (5.27) |
| 27 | $t^{-1}\varphi(u-\beta\ln t, v)$ | (5.28) |
| 28 | $t^{-1}\varphi(v, w)$ | (5.29) |
| 29 | $\varphi(u-t, v-\alpha^{-1}(x-t^2/2))$ | (5.30) |
| 30 | $\varphi(x-t^2/2, u-t)$ | (5.31) |
| 31 | $\varphi(u, v-x)$ | (5.32) |
| 32 | $\varphi(x, u)$ | (5.33) |
| 33 | $\varphi(u-t, v)$ | (5.34) |
| 34 | $\varphi(t, u-x/t)$ | (5.35) |
| 35 | $\varphi(t, w+ut-x)$ | (5.36) |
| 36 | $\varphi(t, u)$ | (5.37) |
| 37 | $\varphi(t, (\boldsymbol{v}-\boldsymbol{x}/t)^2)$ | (6.9) |
| 38 | $\varphi(t, Q)$ | S (6.10) |

Below we restrict our consideration to the Lie algebra $L_{11}(X)$ admissible by the BE (1) with an arbitrary cross section $\sigma$. An application of the optimal system of subalgebras of $L_{11}(Y)$ [6] for constructing invariant solutions of the Boltzmann equation and EGD-system is different. It is connected with different numbers of independent variables and unknown functions. In our case a study of the optimal system [6] gives 11 different classes of invariant solutions with one independent variable and 38 with two independent variables. Their functional expressions are presented in tables 1 and 2.

In the second column of the tables representations of H-solutions are given. In the last column pair index $(m.i)$ means a representative of the optimal subalgebras system, $m$ is a dimension of the corresponding subalgebra and $i$ is its number in table 6 of [8]. In addition, S means that a given representation should be considered in spherical coordinates $(r, \varphi, \theta, U, V, W)$, where $Q = \sqrt{U^2 + V^2 + W^2}$ and $q = \sqrt{V^2 + W^2}$. C corresponds to cylindrical coordinates $(x, r, \theta, u, V, W)$, where also $q = \sqrt{V^2 + W^2}$. Other representations are considered in Cartesian coordinates with $q = \sqrt{v^2 + w^2}$ and $Q = \sqrt{u^2 + v^2 + w^2}$. The $\alpha, \beta, \varepsilon$ are arbitrary constants.

It should be noted that for many representations from tables 1 and 2 some H-solutions either do not exist or do not have a physical meaning. For some of these it can be immediately seen. In the general case corresponding factor equations must be obtained from the BE (1) and studied. However, unlike the EGD-system (4), obtaining factor equations from the BE with complicated collision integral (2) is rather difficult. As an example of the calculation difficulties a factor equation for H-solution (38) in table 2 derived in [18] for the BE (1) with hard sphere molecules ($\sigma_0 = \text{const}$) can be presented:

$$
\begin{aligned}
\frac{\partial f(t, Q)}{\partial t} = {} & 16\frac{\pi^2\sigma_0}{Q}\int_0^Q \int_{\sqrt{Q^2-P^2}}^Q f(t, P) f(t, R)\sqrt{P^2 + R^2 - Q^2}\, P R \,\mathrm{d}P \,\mathrm{d}R \\
& +16\pi^2\sigma_0\left(\int_Q^\infty f(t, P) P \,\mathrm{d}P\right)^2 \\
& +32\frac{\pi^2\sigma_0}{Q}\int_0^Q f(t, P) P^2 \,\mathrm{d}P \int_Q^\infty f(t, P) P \,\mathrm{d}P \\
& -\frac{4}{3}\frac{\pi^2\sigma_0}{Q} f(t, Q)\int_0^\infty f(t, P)[(Q + P)^3 - |Q - P|^3] P \,\mathrm{d}P.
\end{aligned}
$$

*Conclusion*

Here only one applied consequence of the proved isomorphism was used. Further results on the invariant solutions of the obtained BE using the optimal subalgebras systems will be published elsewhere. In the future, investigations of deeper connections between H-solutions of the BE and of the EGD-system will be conducted.

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